

Least Recently Used caches under the Shot Noise Model

Original

Least Recently Used caches under the Shot Noise Model / Leonardi, Emilio; Torrisi, G. L.. - ELETTRONICO. - (2015), pp. 2281-2289. (Intervento presentato al convegno Infocom 2015 tenutosi a Hong Kong nel April 2015) [10.1109/INFOCOM.2015.7218615].

Availability:

This version is available at: 11583/2583367 since: 2015-08-28T10:10:39Z

Publisher:

Published

DOI:10.1109/INFOCOM.2015.7218615

Terms of use:

openAccess

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

Least Recently Used caches under the Shot Noise Model

Emilio Leonardi, Giovanni Luca Torrisi

Abstract—In this paper we develop an analytical framework, based on the Che approximation [2], for the analysis of Least Recently Used (LRU) caches operating under the Shot Noise requests Model (SNM). The SNM was recently proposed in [10] to better capture the main characteristics of today Video on Demand (Vod) traffic. In this context, the Che approximation is derived as the application of a mean field principle to the cache eviction time. We investigate the validity of this approximation through an asymptotic analysis of the cache eviction time. Particularly, we provide a large deviation principle and a central limit theorem for the cache eviction time, as the cache size grows large. Furthermore, we obtain a non-asymptotic analytical upper bound on the error entailed by Che’s approximation of the hit probability.

I. INTRODUCTION

Recently the performance of caching systems, a very traditional and widely studied topic in computer science, has attracted again a significant bunch of attention by the networking research community. This interest revival is due to the important role that caches play in today Internet content distribution systems. Massive Content Delivery Networks (MCDN) represent the standard solution adopted by content and network providers to reach large populations of geographically distributed users in an effective way [11]. Indeed, MCDN permit providers to cache contents close to the users, achieving the twofold goal of reducing network traffic while minimizing the latency suffered by users.

The pervasive presence of caching systems in the Internet is not limited only to existing content delivery networks, as consequence of the gradual shift from the traditional host-to-host communication paradigm to the new host-to-content. A revolutionary Information Centric Network architecture has been recently proposed to fit better the present and future (according to predictions) traffic characteristics [12]. In this architecture, caching becomes an ubiquitous functionality available at each router.

Unfortunately, the performance evaluation of caching systems is very hard, as the computational cost to analyze the behavior of a cache is exponential in both the cache size and the number of contents [13], [14]. For this reason, the effort of the research community has mainly focused on the development of accurate and computationally efficient approximate techniques

for the analysis of caching systems, under various traffic conditions. Che’s approximation [2], proposed for the analysis of Least Recently Used (LRU) caches under the Independent Reference Model (IRM), has emerged as one of the most powerful methods to obtain accurate estimates of the hit probability at limited computational costs [1], [7], [15]. The main idea of this technique is to summarize the response of a cache to the requests arriving for any possible content by a single primitive quantity, which is assumed to be deterministic and the same for any content. This approximation simplifies the analysis of caching systems because it allows to decouple the dynamics of different contents. In particular, in [7] the Che approximation for LRU caches under the IRM found a theoretical justification. Indeed, under mild conditions on the content popularity distribution, the coefficient of variation of the random variable representing the cache eviction time of a fixed content tends to vanish as the cache size grows large. Furthermore, the dependence of the cache eviction time on the specific content considered becomes negligible as the catalogue size grows large.

1) *Paper’s contribution:* In this paper we extend the mathematical analysis of Che’s approximation to LRU caches operating under the Shot Noise Model (SNM) [10]. This model provides a simple, flexible and accurate description of the temporal locality found e.g. in Video on Demand (VoD) traffic, capturing today traffic characteristics in a more natural and precise way than traditional traffic models. We investigate the validity of Che’s approximation by means of an asymptotic analysis of the cache eviction time. Specifically, we provide a large deviation principle and a central limit theorem for the cache eviction time, as the cache size grows large. Furthermore, to the best of our knowledge, we give for the first time a non-asymptotic analytical upper bound on the error entailed by Che’s approximation of the hit probability. Our results show that the Che approximation is a provable, highly accurate and scalable tool to assess the performance of LRU caching systems under the SNM.

II. PRELIMINARIES

We consider a cache, whose size (or capacity), expressed in number of objects (or contents), is denoted by C . Note that the storage capacity of a cache is typically measured in bytes, rather than objects, however in the modeling literature it is common to express it in number of objects, to avoid the additional complexity related to heterogeneous object sizes. This assumption is not

E. Leonardi is with Dipartimento di Elettronica e Telecomunicazioni, Politecnico di Torino, Italy, email: leonardi@polito.it: G. L. Torrisi is with CNR-IAC, Italy email: g.torrisi@iac.rm.cnr.it.

particularly restrictive since any object can be split into constant-size pieces (chunks) which are independently cached.

The cache is fed by an exogenous arrival process of objects' requests generated by users. Requests which find the object in the cache are said to produce a hit, whereas requests that do not find the object in the cache are said to produce a miss. The main performance index is the hit probability, which is the fraction of the requests producing a hit.

The miss stream of a cache, i.e. the process of requests which are not locally satisfied by the cache, is forwarded to either other caches or to a common repository containing all the objects, i.e. the entire objects' catalogue. In the literature it is common to neglect all propagation delays, such as the delays produced in the network by requests which occur in negligible time and the delays necessary to possibly insert an object in the cache.

In this paper we focus on caches implementing the LRU policy: upon the arrival of a request, an object not already stored in the cache is inserted into it. If the cache is full, to make room for a new object the least recently used item is evicted, i.e. the object which has not been requested for the longest time is expunged from the cache.

A. Traffic models

In literature several models have been proposed to describe the process of requests arriving at a cache. The simplest and still most widely adopted is certainly the IRM [3], which makes the following two fundamental assumptions: *i*) The catalogue consists of a fixed number of objects, which does not change over the time; *ii*) The process of requests of a given object is modeled by a homogeneous Poisson process.

As a consequence, the IRM completely ignores all temporal correlations in the sequence of requests and does not take into account a key feature of real traffic referred to as temporal locality, which means that if an object is requested at a given time, then it is more likely that the same object will be requested again in the near future. It is well-known that the temporal locality has a beneficial effect on the cache performance, as it increases the hit probability [3].

Several extensions of the IRM have been proposed to incorporate the temporal locality into the traffic model. Existing generalizations [1], [3], [8] typically assume that the process of requests is time-stationary, usually either a renewal process or a Markov or a semi-Markov modulated Poisson process. However, these models do not capture the kind of temporal locality usually encountered in traces related to distribution systems such as VoD. Indeed, even if the above mentioned models describe accurately the short-term correlations in the time scale of content inter-request times, they do not easily capture the macroscopic long-term effects related to content popularity dynamics. These traffic

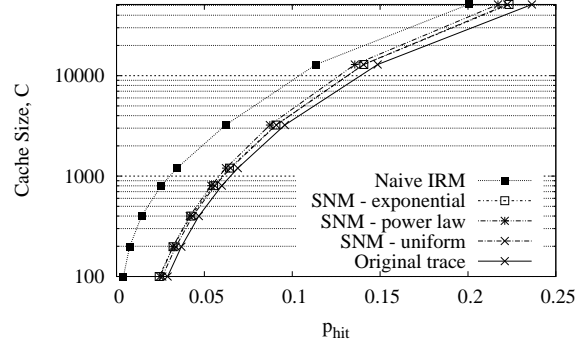


Fig. 1. Hit probability vs cache size, resulting from feeding an LRU cache by: the original YouTube trace, the fitted multi-class SNM, a reshuffled trace analogous to the application of the IRM model.

characteristics are instead well described by the SNM proposed in [10].

The basic idea of the SNM is to represent the requests' process as the superposition of many independent processes, each one referring to a specific object. The requests' process of a fixed content m is specified by two physical (random) parameters: ξ_m and Z_m . ξ_m represents the time instant at which the content enters the system (i.e. it becomes available to the users); Z_m is an attribute of the content m , which summarizes its main characteristics (content type, volume, etc.). Given the couple (ξ_m, Z_m) , the requests' process of content m is assumed to be a non-homogeneous Poisson process with an assigned intensity function $h(\cdot - \xi_m, Z_m)$. Here h is a non-negative function, which has to be interpreted as the popularity profile of content m . On this regard, we recall that recent experimental works [18], [19], [20] have shown that on-line contents (e.g. videos) can be clustered in a surprisingly small number of classes, each one exhibiting a typical temporal profile.

As a proof of concepts, Figure 1 reports the cache size needed to achieve a prefixed hitting probability for an LRU cache fed by a real trace of YouTube video requests, which was kindly provided to us by the authors of [10]. In [10], this trace has been fitted by an SNM in which contents are clustered in 4 classes, each one associated to a particular normalized temporal popularity profile. Note that, by wisely calibrating the parameters, the results obtained exploiting the SNM are in a very good agreement with those one obtained by directly feeding the cache with the experimental trace. Figure 1 reports also a curve named "Naive IRM", corresponding to a random permutation of the requests contained in the original trace: by so doing we wash out the temporal correlations existing in the original trace.

III. CACHE ANALYSIS UNDER THE SHOT NOISE MODEL

The SNM is a dynamical model according to which every content m is introduced into the system (i.e. the catalogue) at time ξ_m and then requested by the users according to a prefixed popularity profile. We assume that the set of times $N \equiv \{\xi_m\}_{m \geq 1}$ at which contents become available is distributed according to a homogeneous Poisson process on \mathbb{R} with intensity $\lambda > 0$. Here, $\{\xi_m\}_{m \geq 1}$ is supposed to be an unordered set of times. We suppose that, after the introduction into the catalogue of the content m , the requests for this content arrive at the cache according to a Cox process $N^{(m)} \equiv \{T_n^{(m)}\}_{n \geq 1}$ on \mathbb{R} whose stochastic intensity $\{\lambda_m(t)\}_{t \in \mathbb{R}}$ is defined by

$$\lambda_m(t) := h(t - \xi_m, Z_m).$$

We assume that $\{Z_m\}_{m \geq 1}$ is a sequence of independent and identically distributed random variables, independent of $\{\xi_m\}_{m \geq 1}$, with values on some measurable space (E, \mathcal{E}) . Furthermore, we suppose that $h : \mathbb{R} \times E \rightarrow [0, \infty)$ is a measurable non-negative function such that $h(t, z) = 0$ for any $t < 0$ and $z \in E$. For any $z \in E$, the function $h(\cdot, z)$ has to be interpreted as the instantaneous popularity profile of the content we are considering. Finally, we suppose that, for any $m \geq 1$, $T_1^{(m)} < T_2^{(m)} < \dots$ almost surely and we assume that the Cox processes $\{N^{(m)}\}_{m \geq 1}$ are independent, given $\{(\xi_m, Z_m)\}_{m \geq 1}$.

A. Formal definition of the cache eviction time

We denote by m_0 a tagged content introduced into the catalogue at the deterministic time $x_{m_0} \in \mathbb{R}$. Moreover, we denote by $\mathbf{X}(t)$, $t > 0$, the number of contents different from m_0 that have been requested in the time interval $[0, t]$, i.e.

$$\mathbf{X}(t) = \sum_{m \neq m_0} \mathbf{1}\{m \text{ requested in } [0, t]\} \mathbf{1}\{\xi_m \in (-\infty, t]\}.$$

Observe that the distribution of $\mathbf{X}(t)$ is invariant with respect to a rigid shift of the observation window $[0, t]$. Throughout this paper we shall consider the random variable

$$X_{m_0}(t) := \mathbf{X}(t) | \xi_{m_0} = x_{m_0}, \quad t > 0,$$

which plays an important role in the dynamics of an LRU cache because the cache eviction time may be expressed in terms of $X_{m_0}(t)$. Indeed, under the LRU replacement policy, assuming that the content m_0 is requested at time $t = 0$, we have that it is expunged from the cache (provided it has not requested in the meanwhile) as soon as the C th content, different from m_0 , is requested since the last request for m_0 . So, under the LRU replacement policy, the so-called cache eviction time for the content m_0 is given by the random variable

$$T_C(m_0) := \inf\{t > 0 : X_{m_0}(t) = C\}.$$

B. The distribution of $X_{m_0}(t)$

Define the quantity

$$g(t) := \int_0^\infty \mathbb{E} \left[1 - e^{-\int_{\max\{u-t, 0\}}^u h(s, Z_1) ds} \right] du, \quad t > 0.$$

The following proposition holds.

Proposition 3.1: If $g(t) < \infty$, then the random variable $X_{m_0}(t)$ is Poisson distributed with mean $\lambda g(t)$.

The proof is reported in the appendix C. Note that the condition $g(t) < \infty$ is fairly general: for example, it is satisfied whenever the popularity profile is of the form $zh(\cdot)$, with $h(\cdot)$ integrable, and $\mathbb{E}[Z_1] < \infty$.

In the context of an LRU cache under the SNM, Che's approximation consists in replacing the cache eviction time $T_C(m_0)$ by the deterministic constant

$$t_C(m_0) := \inf\{t > 0 : \mathbb{E}[X_{m_0}(t)] = C\}.$$

Note that, if $g(t) < \infty$ for any $t > 0$, then by Proposition 3.1 we have

$$t_C(m_0) = \inf\{t > 0 : \lambda g(t) = C\},$$

and so when $g : (0, \infty) \rightarrow (0, \infty)$ is in addition a strictly increasing function (again conditions for this to happen are fairly general and cover all the cases of practical interest) we deduce

$$t_C(m_0) = g^{-1}(C/\lambda). \quad (1)$$

Since the law of $X_{m_0}(t)$ (and therefore of $T_C(m_0)$) does not depend on m_0 , hereafter we simply write $X(t)$, T_C and t_C in place of $X_{m_0}(t)$, $T_C(m_0)$ and $t_C(m_0)$.

C. Asymptotic analysis of T_C

In this section we investigate the validity of Che's approximation for large values of C . We shall do this by analyzing the behavior of T_C as $C \uparrow \infty$. Intuitively, Che's approximation finds a theoretical justification if we may show that, as $C \uparrow \infty$, $T_C/t_C \rightarrow 1$ almost surely. This is indeed achieved in Proposition 3.2. Proposition 3.3 below and the subsequent computation provide some estimates on the probability that T_C deviates from its most probable value t_C , as C grows large. Finally, the Gaussian approximation for T_C in Proposition 3.4 gives some insights on how the random variable $T_C/t_C - 1$ scales to zero, as $C \rightarrow \infty$.

1) Law of the large numbers and tail estimates for the cache eviction time: The following law of large numbers holds.

Proposition 3.2: Provided that $g : (0, \infty) \rightarrow (0, \infty)$ is strictly increasing and that $g, g^{-1} : (0, \infty) \rightarrow (0, \infty)$ are bijective and continuous (i.e. g is a homeomorphism of $(0, \infty)$). Then

$$\lim_{C \rightarrow \infty} \frac{T_C}{t_C} = 1, \quad \text{almost surely.} \quad (2)$$

Next proposition and the subsequent computation give asymptotic estimates on the probability that the cache eviction time deviates from its most probable value.

Proposition 3.3: Under the assumptions of Proposition 3.2, we have

$$\lim_{C \rightarrow \infty} \frac{1}{C} \log \mathbb{P}(T_C > g^{-1}(Cx_r)) = -(\lambda x_r - 1 - \log(\lambda x_r)), \quad \forall x_r > 1/\lambda \quad (3)$$

and

$$\lim_{C \rightarrow \infty} \frac{1}{C} \log \mathbb{P}(T_C \leq g^{-1}(Cx_l)) = -(\lambda x_l - 1 - \log(\lambda x_l)), \quad \forall x_l \in (0, 1/\lambda). \quad (4)$$

Note that, under the foregoing assumptions, by (3) and (4) (and the fact that $t_C = g^{-1}(C/\lambda)$) it follows that for an arbitrarily small $\delta \in (0, 1)$, for any $\varepsilon > 0$ there exists C_ε so that for any $C > C_\varepsilon$

$$\begin{aligned} e^{-C(I(g(t_C(1+\delta)))/C)+\varepsilon} &\leq \mathbb{P}(T_C > t_C(1+\delta)) \\ &\leq e^{-C(I(g(t_C(1+\delta)))/C)-\varepsilon} \end{aligned}$$

and

$$\begin{aligned} e^{-C(I(g(t_C(1-\delta)))/C)+\varepsilon} &\leq \mathbb{P}(T_C \leq t_C(1-\delta)) \\ &\leq e^{-C(I(g(t_C(1-\delta)))/C)-\varepsilon}. \end{aligned}$$

Here $I(x) := \lambda x - 1 - \log(\lambda x)$.

The proofs of the above propositions are given in the appendix B. As we shall see, these results are obtained as a consequence of a general large deviation principle for the process $\{g(T_C)/C\}_{C \geq 1}$; the reader is directed to [5] for an introduction on the large deviations theory.

As a simple and meaningful example for which deriving an explicit expression of g , let us consider the case:

$$h(t, z) := \frac{z}{L} \mathbb{1}_{[0, L]}(t), \quad \text{for some constant } L > 0 \quad (5)$$

(which is the shape used to obtain the results of Fig. 1) and Z_1 with law supported on $E = (a, \infty)$, $a > 0$. A possible choice for the law of Z_1 is the Pareto distribution with finite mean, which is closely related to the Zipf's law, (see Section III-G). With this choice of h and Z_1 we have (see [21] for the details):

$$\begin{aligned} g(t) &= 2t + (L - t) \mathbb{E} \left[1 - e^{-\frac{Z_1}{L}t} \right] \\ &\quad - 2 \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-\frac{Z_1}{L}t} \right) \right], \quad t \in (0, L] \end{aligned}$$

and

$$\begin{aligned} g(t) &= 2L + (t - L) \mathbb{E} \left[1 - e^{-Z_1} \right] \\ &\quad - 2 \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-Z_1} \right) \right], \quad t > L \end{aligned}$$

Of course the derived expression of g satisfies all the assumptions of Proposition 3.2.

2) *Normal approximation of the cache eviction time:* Hereafter, we denote by $\mathcal{N}(0, 1)$ a standard normal random variable and by $\xrightarrow{\text{law}}$ the convergence in distribution. The following central limit theorem holds.

Proposition 3.4: Assume that $g : (0, \infty) \rightarrow (0, \infty)$ is a strictly increasing function and suppose that

\exists a positive function f such that $\exists \lim_{y \rightarrow \infty} f(y) \in [0, \infty]$ and

$$\lim_{y \rightarrow \infty} \frac{g(y) - g(y + xf(y))}{\sqrt{g(y + xf(y))}} = -\frac{x}{\sqrt{\lambda}}. \quad (6)$$

Then

$$\frac{T_C - t_C}{f(t_C)} \xrightarrow{\text{law}} \mathcal{N}(0, 1), \quad \text{as } C \rightarrow \infty. \quad (7)$$

Note that from (2) and (7), we have $f(x)/x \rightarrow 0$, as $x \rightarrow \infty$. Therefore, from (7) we deduce that the random variable $\frac{T_C}{t_C} - 1$ scales to zero as $f(t_C)/t_C$ as C grows large. For example, if we consider a popularity profile $h(\cdot, z)$ as in (5) we have (see [21])

$$f(x) := \sqrt{\frac{x}{\lambda \mathbb{E}[1 - e^{-Z_1}]}}$$

and so $\frac{T_C}{t_C} - 1$ scales to 0 as $\frac{1}{\sqrt{t_C}}$, as $C \rightarrow \infty$. We refer the reader to the appendix for the proof of Proposition 3.4.

D. The "in" probability

Under the Che approximation, the probability of finding the tagged content m_0 in the cache after an interval of length $t - x_{m_0}$ from its introduction into the catalogue (which happens at time x_{m_0}) is equal to

$$\begin{aligned} p_{\text{in, Che}}^{(t-x_{m_0})}(z_{m_0}, t_C) &:= \\ \mathbb{P}(N^{(m_0)}((t - t_C, t]) \geq 1 \mid (\xi_{m_0}, Z_{m_0}) = (x_{m_0}, z_{m_0})) \\ &= 1 - e^{-\int_{t-t_C}^t h(u-x_{m_0}, z_{m_0}) du} \\ &= 1 - e^{-\int_{t-x_{m_0}-t_C}^{t-x_{m_0}} h(u, z_{m_0}) du}, \end{aligned} \quad (8)$$

where $N^{(m)}((t - t_0, t])$ denotes the number of points $\{T_n^{(m)}\}_{n \geq 1}$ in the time interval $(t - t_0, t]$. Without relying on the Che approximation, the probability of finding the content m_0 in the cache after a time interval of length $t - x_{m_0}$ from its introduction into the catalogue is instead

$$\begin{aligned} p_{\text{in}}^{(t-x_{m_0})}(z_{m_0}, T_C) &:= \\ \mathbb{P}(N^{(m_0)}((t - T_C, t]) \geq 1 \mid (\xi_{m_0}, Z_{m_0}) = (x_{m_0}, z_{m_0}), T_C) \\ &= p_{\text{in, Che}}^{(t-x_{m_0})}(z_{m_0}, T_C). \end{aligned}$$

E. The "hit" probability

Under the Che approximation, the probability that the tagged content m_0 is found in the cache by an arriving

request at time t is equal to

$$\begin{aligned} p_{\text{hit,Che}}^{(t-x_{m_0})}(z_{m_0}, t_C) &:= \\ \mathbb{P}\left(\sum_{T_n^{(m_0)} \in N^{(m_0)} \setminus \{t\}} \mathbf{1}_{(t-t_C, t]}(T_n^{(m_0)}) \geq 1 \mid t \in N^{(m_0)}, \right. \\ &\quad \left. (\xi_{m_0}, Z_{m_0}) = (x_{m_0}, z_{m_0})\right) = \\ \mathbb{P}\left(N^{(m_0)}((t-t_C, t]) \geq 1 \mid (\xi_{m_0}, Z_{m_0}) = (x_{m_0}, z_{m_0})\right) \\ &= p_{\text{in,Che}}^{(t-x_{m_0})}(z_{m_0}, t_C), \end{aligned}$$

where the first equality is a consequence of the Slivnyak theorem (see e.g. [4]). Under the Che approximation, we define the hit probability as the ratio between the average rate at which hits of the tagged content m_0 occur and the average rate at which requests of the tagged content m_0 are observed, i.e.

$$p_{\text{hit,Che}}(t_C) := \frac{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0}) p_{\text{in,Che}}^{(t-\xi_{m_0})}(Z_{m_0}, t_C)]}{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0})]},$$

with the convention $0/0 = 0$. Note that the probability $p_{\text{hit,Che}}(t_C)$ does not depend on m_0 and t . Indeed, for an arbitrary s we have

$$\begin{aligned} &\frac{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0}) p_{\text{in,Che}}^{(t-\xi_{m_0})}(Z_{m_0}, t_C) \mathbf{1}\{s < \xi_{m_0} < t\}]}{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0}) \mathbf{1}\{s < \xi_{m_0} < t\}]} \\ &= \frac{(t-s)^{-1} \int_s^t \mathbb{E}[h(t-u, Z_1) p_{\text{in,Che}}^{(t-u)}(Z_1, t_C)] du}{(t-s)^{-1} \int_s^t \mathbb{E}[h(t-u, Z_1)] du} \\ &= \frac{\int_s^t \mathbb{E}[h(t-u, Z_1) p_{\text{in,Che}}^{(t-u)}(Z_1, t_C)] du}{\int_s^t \mathbb{E}[h(t-u, Z_1)] du}, \end{aligned}$$

and so letting s tend to $-\infty$ we deduce

$$p_{\text{hit,Che}}(t_C) = \frac{\int_0^\infty \mathbb{E}[h(u, Z_1) p_{\text{in,Che}}^{(u)}(Z_1, t_C)] du}{\int_0^\infty \mathbb{E}[h(u, Z_1)] du}.$$

Without relying on the Che approximation, we define the probability that the tagged content m_0 is found in the cache by an arriving request at time t by

$$p_{\text{hit}}(T_C) := \frac{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0}) p_{\text{in,Che}}^{(t-\xi_{m_0})}(Z_{m_0}, T_C) \mid T_C]}{\mathbb{E}[h(t - \xi_{m_0}, Z_{m_0})]}.$$

Arguing as above, one may easily check that

$$p_{\text{hit}}(T_C) = \frac{\int_0^\infty \mathbb{E}[h(u, Z_{m_0}) p_{\text{in,Che}}^{(u)}(Z_{m_0}, T_C) \mid T_C] du}{\int_0^\infty \mathbb{E}[h(u, Z_1)] du},$$

and so, being Z_{m_0} and T_C independent, the hitting probability is

$$p_{\text{hit}} = \int_{[0, \infty)} p_{\text{hit,Che}}(\theta) \mathbb{P}_{T_C}(d\theta),$$

where \mathbb{P}_{T_C} denotes the law of T_C and

$$p_{\text{hit,Che}}(\theta) := \frac{\int_0^\infty \mathbb{E}[h(u, Z_1) p_{\text{in,Che}}^{(u)}(Z_1, \theta)] du}{\int_0^\infty \mathbb{E}[h(u, Z_1)] du}.$$

By using the above relations and classical estimates for the tail of a Poisson distribution, we can evaluate the error committed by approximating p_{hit} with $p_{\text{hit,Che}}(t_C)$. The following proposition holds.

Proposition 3.5: If $g : (0, \infty) \rightarrow (0, \infty)$ is strictly increasing, then, for any $\delta \in (0, 1)$ and $C > 0$, we have

$$\begin{aligned} |p_{\text{hit}} - p_{\text{hit,Che}}(t_C)| &\leq \\ &\exp(-\lambda g(t_C(1-\delta))H(C/\lambda g(t_C(1-\delta)))) + \\ &\exp(-\lambda g(t_C(1+\delta))H(C/\lambda g(t_C(1+\delta)))) + \\ &\max_{\theta \in \{t_C(1-\delta), t_C(1+\delta)\}} |p_{\text{hit,Che}}(\theta) - p_{\text{hit,Che}}(t_C)|, \end{aligned}$$

where $H(x) = 1 - x + x \log x$, $x > 0$.

The proof is reported in the appendix. Proposition 3.5 provides estimates on the error entailed by the Che approximation, permitting an assessment of its accuracy in different scenarios. As we will see in Section III-G, in most cases exploiting Proposition 3.5, we can show that Che's approximation leads to surprisingly accurate predictions of caching performance.

F. Some closed form estimates of the hit probability

As shown by several recent experimental works, typical available video contents (such as YouTube contents) exhibits few typical normalized temporal popularity profiles, each profile corresponding to a large class of contents with similar characteristics (e.g. contents in the same YouTube category) [18], [19], [20]. Hence, restricting the analysis to a single class m of contents, we may assume that: *i*) Z_m represents the content volume, i.e. the total number of requests it typically attracts; *ii*) all contents of the class exhibit the same normalized popularity profile.

As a consequence, with a little abuse of notation, the popularity profile function may be taken of the form

$$h(t, z) = zh(t), \quad t > 0, z \in E \subseteq (a, \infty), a > 0 \quad (9)$$

being $h(\cdot)$ a nonnegative function such that $\int_0^\infty h(t) dt = 1$. In such scenarios, assuming $\mathbb{E}[Z_1] < \infty$, we have

$$g(t) = \int_0^\infty \left[1 - \phi_{Z_1} \left(- \int_{\max\{u-t, 0\}}^u h(s) ds \right) \right] du \quad (10)$$

and

$$\begin{aligned} p_{\text{hit,Che}}(\theta) &= \\ 1 - (\mathbb{E}[Z_1])^{-1} \int_0^\infty h(u) \phi'_{Z_1} \left(- \int_{u-\theta}^u h(s) ds \right) du, \end{aligned} \quad (11)$$

where $\phi_{Z_1}(\theta) := \mathbb{E}[\exp(\theta Z_1)]$, $\theta \in \mathbb{R}$; $\phi'_{Z_1}(\theta)$ is the first derivative of $\phi_{Z_1}(\theta)$.

Relations (10) and (11) provide a computationally efficient tool to estimate the hit probability of LRU caches under the SNM. Indeed, we may estimate t_C by numerically inverting (10) and using the relation

$t_C = g^{-1}(C/\lambda)$. Replacing θ in (11) with such estimate of t_C , we finally deduce an estimate of the hit probability under the Che approximation. Expressions similar to (11) and (10) have been obtained in [?] by applying heuristic arguments, which are not mathematically well founded. Here, for the first time, we propose a correct mathematical procedure to derive and justify the Che's approximation for LRU caches.

Assuming $\mathbb{E}[Z_1^2] < \infty$ and $L := (\int_0^\infty h^2(t)dt)^{-1} \in (0, \infty)$, by the Taylor formula and (11) we have the following expansion of $p_{\text{hit,Che}}(\cdot)$ in a neighborhood of the origin:

$$p_{\text{hit,Che}}(\theta) = \frac{\theta \mathbb{E}[Z_1^2]}{L \mathbb{E}[Z_1]} + o(\theta), \quad \text{as } \theta \rightarrow 0. \quad (12)$$

Applying again the Taylor formula and using the relation $C = \lambda g(t_C)$, we deduce (note that $t_C, g(t_C) \rightarrow 0$, as $\frac{C}{\lambda} \rightarrow 0$)

$$C = \lambda \mathbb{E}[Z_1] t_C + o(t_C), \quad \text{as } \frac{C}{\lambda} \rightarrow 0. \quad (13)$$

Combining (12) and (13), we obtain the following approximation for the hit probability under the Che approximation, when the cache size is small relatively to the effective catalogue size (i.e., $\frac{C}{\lambda}$ small):

$$p_{\text{hit,Che}}(t_C) \approx \frac{C \mathbb{E}[Z_1^2]}{\lambda L \mathbb{E}^2[Z_1]}, \quad \text{as } \frac{C}{\lambda} \rightarrow 0. \quad (14)$$

From (14) we deduce the following insights when the cache size is small relatively to the effective catalogue size:

- i) The hit probability depends on the popularity profile only through L . This is an important insensitivity property of the system.
- ii) The cache performance depends on the content volume distribution (i.e. the law of Z_1) only through the ratio of the first two moments.
- iii) The hit probability increases linearly with the cache size.

Finally, we note that from (11), we can obtain a closed form expression for the asymptotic hit probability when C grows large, under Che's approximation. Indeed,

$$\begin{aligned} p_{\text{hit,Che},\infty} &:= \lim_{C \rightarrow \infty} p_{\text{hit,Che}}(t_C) \\ &= 1 - \frac{1 - \phi_{Z_1}(-1)}{\mathbb{E}[Z_1]} \end{aligned} \quad (15)$$

where the last expression is obtained by letting $\theta \rightarrow \infty$ in (11) and then applying the following change of variable inside the integral: $v = \int_0^u h(u)du$.

G. Numerical Results

The goal of this section is two-fold. On the one hand, we assess the accuracy of the Che approximation for the evaluation of the cache hit probability. On the other hand, we exploit the insights gained from the analytical predictions to better understand the performance of

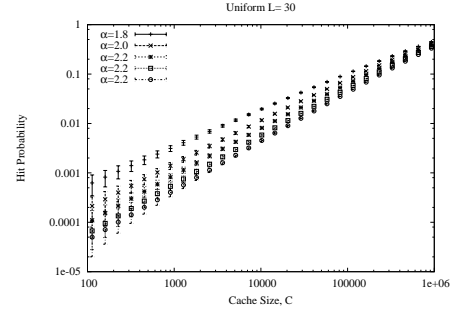


Fig. 2. p_{hit} vs cache size for different values of the exponent $\alpha > 1$ and a fixed content life-span $L = 30$.

caching systems under the shot noise traffic model. Our study reveals that Che's approximation can be effectively applied for system design and optimization since it provides accurate predictions of the performance of LRU caches under fairly realistic traffic patterns at low computational cost.

Assuming a popularity profile of the form (9), we take the arrival rate of new contents λ equal to 100.000 units per day and assume that volumes (Z 's), which represent the average number of requests attracted by contents, follow a Pareto distribution with probability density

$$f_{Z_1}(z) = \alpha a^\alpha / z^{1+\alpha}, \quad z \geq a > 0.$$

The choice of a Pareto distribution has the following motivation. First previous works have shown that the aggregate requests attracted by many types of contents (including popular movies or user-generated videos) over long time periods are well described by the Zipf's law [7]. Second, a Zipf-like empirical distribution is obtained when a large collection of quantities independently drawn from a Pareto distribution, are sorted in decreasing order.

Unless differently stated, in the numerical illustrations below, we take $\alpha > 1$, $\mathbb{E}[Z_1] = \frac{\alpha a}{\alpha - 1} = 3$ and $h(t) = \frac{1}{L} \mathbf{1}\{0 \leq t < L\}$ corresponding to a uniform popularity profile with life-span L .

Figure 2 reports the hit probability, as predicted by the Che approximation, vs the cache size for different values of the exponent α . The content life-span L has been fixed equal to 30 for all contents. For each approximate estimate, Figure 2 reports also the interval in which the exact value of the hit probability falls as given by Proposition 3.5. All computations have been carried out while guaranteeing relative numerical errors smaller than 10^{-2} .

Note that in all cases of practical relevance (i.e., for values of the hit probability exceeding 0.01) Che's approximation leads to negligible errors. The surprisingly good degree of accuracy entailed by Che's approximation, which has been already experimentally (i.e., against simulations) observed by several authors [7], [15], is now confirmed even for the SNM. Note also that, as expected, the distribution of Z_1 plays a

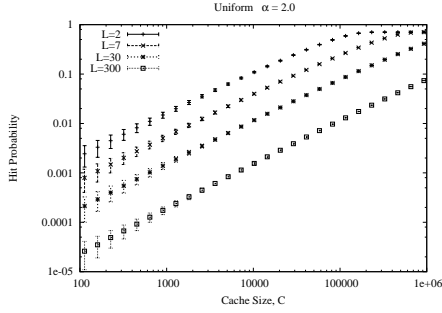


Fig. 3. p_{hit} vs cache size for different values of the content life-span L . In all cases $\alpha = 2$.

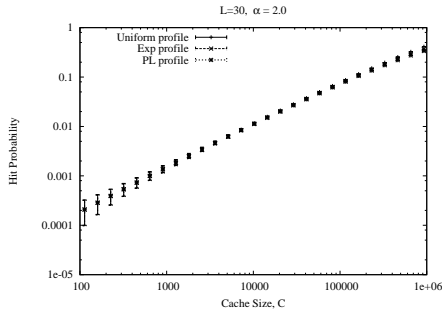


Fig. 4. p_{hit} vs cache size for different temporal profiles $h(\cdot)$. In all cases $\alpha = 2$ and $L = 30$.

role: caching performance improves as the parameter α decreases. However, the impact of the specific α appears to be fairly limited as long as $\alpha > 2$ (i.e. the variance of Z_1 is finite). This is in sharp contrast to previous results for the IRM, where the Zipf's exponent has a huge impact on the cache performance [7].

Figure 3 reports the hit probability vs the cache size for different values of the life-span L . The parameter α has been fixed equal to 2. Again the errors induced by the Che approximation are negligible for every practical purpose. Note that the content life-span L plays a major role on the caching performance, which significantly benefits from the temporal locality of contents. For a given cache size, the hit probability appears roughly inversely proportional to L .

Figure 4 reports the hit probability vs the cache size for different temporal profiles $h(\cdot)$. The parameters α and L have been fixed equal to 2 and 30, respectively. In particular, the following profiles have been considered: the uniform shaped profile

$$h(t) := \frac{1}{L} \mathbf{1}\{0 \leq t < L\},$$

the exponentially shaped profile

$$h(t) := \frac{1}{2L} \exp(-t/(2L)) \mathbf{1}\{t \geq 0\}$$

and the polynomial tailed profile

$$h(t) := \frac{10}{4L} \left(\frac{5t}{4L} + 1 \right)^{-3} \mathbf{1}\{t \geq 0\}.$$

The impact of the temporal profiles on the cache performance appears fairly marginal both for small caches values and for moderately large values of C . This is in accordance with (14) and (15). Furthermore, in all cases Che's approximation provides very good estimates of the real value of the hit probability.

Finally, we wish to mention that we have run Monte-Carlo simulations for all the cases reported in Figures 2, 3 and 4, using a simulator that was kindly provided to us by the authors of [10]. We set the confidence interval level to 0.99 and the confidence interval half-width to 10 % of the nominal value as long as the nominal value of the hit probability exceeds 0.05 and to 30 % of the nominal value in the other cases. In all cases we have observed a very good agreement between simulations and our analysis, with the central value of simulations always falling within the theoretical interval predicted by Proposition 3.5. We remark that the algorithm produced by our analytical results, in terms of CPU time, is up to two orders of magnitude more efficient than the Monte-Carlo approach. The corresponding simulation results have not been reported in the above figures for the sake of figure readability.

IV. FUTURE RESEARCH: NETWORKS OF CACHES

It is known that the analysis of networks of caches, i.e. systems of interconnected caches, is a difficult task. Indeed, the process of requests at every non-ingress cache (i.e. a cache to which miss requests from other caches are forwarded) is not anymore exogenous. In fact, it is a combination of miss streams at other caches (those preceding the considered cache along the requests path routes) and exogenous processes. The characterization of the miss stream of an LRU cache is prohibitive even under the IRM, and this poses serious problems in the analysis of networks of caches. A rather crude approach that has been proposed in [17] for networks of caches under the IRM, consists in approximating the miss stream of a content at a cache with a homogeneous Poisson process whose rate matches the miss stream rate. In this case, however, significant errors may be experienced. A different approach, which has been recently proposed for feed-forward networks of caches (such as networks with linear topologies or trees) operating under classical traffic models, consists in considering the miss stream of an LRU cache under the Che approximation (see [16], [1]). This approach has been experimentally shown to be rather accurate, and we believe that it may be successfully applied even to networks of caches under the SNM. In particular, we believe that the asymptotic analysis of the cache eviction time carried on in this paper may be used to evaluate, as the cache size grows large, the error committed by replacing the exact miss stream with the miss stream based on the Che approximation. We leave this research topic for future work.

REFERENCES

- [1] G. Bianchi et al. Check before storing: what is the performance price of content integrity verification in LRU caching? *ACM Comput. Comm. Rev.*, 43(3):59–67, 2013.
- [2] H. Che, Y. Tung and Z. Wang. Hierarchical Web caching systems: modeling, design and experimental results. *IEEE JSAC*, 20(7):1305–1314, 2002.
- [3] E. Coffman and P. Denning. *Operating Systems Theory*. Englewood Cliffs, New Jersey, 1974.
- [4] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes. Vol. I*. Springer, New York, 2003.
- [5] A. Dembo and O. Zeitouni. *Large Deviation Techniques and Applications*. Springer, New York, 1998.
- [6] N.G. Duffield and W. Whitt. Large deviations of inverse processes with nonlinear scalings. *Ann. Appl. Probab.*, 8(4):995–1026, 1998.
- [7] C. Fricker, P. Robert and J. Roberts. A versatile and accurate approximation for LRU cache performance. *Proceedings of ITC*, Krakow, Poland, 2012.
- [8] K. Kylakoski and J. Virtamo. Cache Replacement Algorithms for the Renewal Arrival Model. *Proc. of Fourteenth Nordic Teletraffic Seminar*, Copenhagen, Denmark, 1998.
- [9] M. Penrose. *Random geometric graphs*. Oxford University Press, New York, 2004.
- [10] S.Traverso et al. Temporal Locality in Today's Content Caching: Why It Matters and How to Model It. *ACM Comput. Comm. Rev.*, 43(5):5–12, 2013.
- [11] W. Jiang et al., Orchestrating massively distributed CDNs. *ACM CoNEXT*, Nice, France, 2013.
- [12] V. Jacobson et al. Networking named content. *ACM CoNEXT*, Rome, Italy, 2009.
- [13] K.W. King. Analysis of paging algorithms. *IFIP Congress*, 1971.
- [14] A. Dan and D. Towsley. An approximate analysis of the LRU and FIFO buffer replacement schemes. *SIGMETRICS Perform. Eval. Rev.*, 18(1):143–152, 1990.
- [15] V. Martina et al. A unified approach to the performance analysis of caching systems. *IEEE INFOCOM*, Toronto, Canada, 2014.
- [16] N.C. Fofack et al. Performance evaluation of hierarchical TTL-based cache networks. *Computer Networks*, 65(2):212–231, 2014.
- [17] E.J. Rosensweig et al. Approximate Models for General Cache Networks. *IEEE INFOCOM*, San Diego, USA, 2010.
- [18] R. Crane and D. Sornette. Robust dynamic classes revealed by measuring the response function of a social system. *PNAS*, 105(41):15649–15653, 2008.
- [19] M. Ahmed et al. A peek into the future: predicting the evolution of popularity in user generated content. *WSDM*, 2013.
- [20] H. Pinto et al. Using early view patterns to predict the popularity of Youtube videos. *WSDM*, 2013.
- [21] Anonymous Tech. Rep. https://www.dropbox.com/x2fgsd5puxphf15/tech_rep.pdf

APPENDIX A

PROOF OF PROPOSITION 3.1

For any $t_0 > t > 0$, we define the "restriction" of $\mathbf{X}(t)$ to contents that have been introduced in the catalogue in the time interval $[t - t_0, t]$ by

$$\mathbf{X}_{t_0}(t) = \sum_{m \neq m_0} \mathbf{1}\{m \text{ requested in } [0, t]\} \mathbf{1}\{\xi_m \in [t - t_0, t]\}.$$

By the Slivnyak theorem (see e.g. [4]), the law of $\{\xi_m\}_{m \neq m_0}$ given the event $\{\xi_{m_0} = x_{m_0}\}$ coincides

with the law of $\{\xi_m\}_{m \geq 1}$ and so, for any $\theta \in \mathbb{R}$,

$$\mathbb{E} \left[e^{\theta \mathbf{X}_{t_0}(t)} \mid \xi_{m_0} = x_{m_0} \right] = \mathbb{E} \left[e^{\theta \tilde{\mathbf{X}}_{t_0}(t)} \right], \quad (16)$$

where

$$\tilde{\mathbf{X}}_{t_0}(t) = \sum_{m \geq 1} \mathbf{1}\{m \text{ requested in } [0, t]\} \mathbf{1}\{\xi_m \in [t - t_0, t]\}.$$

Letting $N([t - t_0, t])$ denote the number of points $\{\xi_m\}_{m \geq 1}$ in the time interval $[t - t_0, t]$ and $N^{(m)}([0, t])$ denote the number of points $\{T_n^{(m)}\}_{n \geq 1}$ in the time interval $[0, t]$, we rewrite $\tilde{\mathbf{X}}_{t_0}(t)$ as

$$\tilde{\mathbf{X}}_{t_0}(t) = \sum_{m=1}^{N([t-t_0, t])} \mathbf{1}\{N^{(m)}([0, t]) \geq 1\}.$$

Since, given ξ_m and Z_m , $N^{(m)}$ is a Poisson process with intensity function $h(\cdot - \xi_m, Z_m)$, we have

$$p_t(\xi_m, Z_m) := \mathbb{P}(N^{(m)}([0, t]) \geq 1 \mid \xi_m, Z_m) = 1 - e^{-\int_0^t h(s - \xi_m, Z_m) ds} = 1 - e^{-\int_{\max\{-\xi_m, 0\}}^{t - \xi_m} h(s, Z_m) ds}.$$

Recalling that, given $\{N([t - t_0, t]) = k\}$, the k points of N on $[t - t_0, t]$ are independent and uniformly distributed over $[t - t_0, t]$ (see e.g. [4]), for any $\theta \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E} \left[e^{\theta \tilde{\mathbf{X}}_{t_0}(t)} \mid N([t - t_0, t]) = k \right] &= \\ \mathbb{E} \left[\prod_{m=1}^k e^{\theta \mathbf{1}\{N^{(m)}([0, t]) \geq 1\}} \mid N([t - t_0, t]) = k \right] &= \\ \prod_{m=1}^k \left(1 + (e^\theta - 1) \frac{1}{t_0} \int_{t-t_0}^t \mathbb{E}[p_t(u, Z_1)] du \right) &= \\ \left[\frac{1}{t_0} \int_{-t_0}^0 \mathbb{E} \left[1 + (e^\theta - 1) p_t(u + t, Z_1) \right] du \right]^k. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[e^{\theta \tilde{\mathbf{X}}_{t_0}(t)} \right] &= e^{-\lambda t_0} \cdot \\ \sum_{k \geq 0} \left[\frac{\lambda}{k!} \int_{-t_0}^0 \mathbb{E} \left[1 + (e^\theta - 1) p_t(u + t, Z_1) \right] du \right]^k &= \\ = \exp \left(\lambda (e^\theta - 1) \int_{-t_0}^0 \mathbb{E} [p_t(u + t, Z_1)] du \right). \end{aligned} \quad (17)$$

The claim follows by (16) and (17), letting t_0 tend to ∞ .

APPENDIX B

PROOFS OF PROPOSITIONS 3.2 AND 3.3

The proofs of Propositions 3.2 and 3.3 are based on the following preliminary lemma which gives the large deviations for the process $\{g(T_C)/C\}_{C \geq 1}$.

Lemma B.1: Under the assumptions of Proposition 3.2, we have that the family of random variables $\{g(T_C)/C\}_{C \geq 1}$ obeys a large deviation principle on

$(0, \infty)$ with speed g and rate function $I(x) := \lambda x - 1 - \log(\lambda x)$ i.e., for all Borel sets $B \subset (0, \infty)$,

$$\begin{aligned} - \inf_{x \in B^\circ} I(x) &\leq \liminf_{C \rightarrow \infty} \frac{1}{C} \log \mathbb{P}(g(T_C)/C \in B) \\ &\leq \limsup_{C \rightarrow \infty} \frac{1}{C} \log \mathbb{P}(g(T_C)/C \in B) \leq - \inf_{x \in \bar{B}} I(x), \end{aligned}$$

where B° denotes the interior of B and \bar{B} denotes the closure of B .

Proof. By Proposition 3.1 and the Gärtner-Ellis theorem (see e.g. [5]) the stochastic process $\{X(t)/g(t)\}_{t \geq 1}$ satisfies a large deviation principle on $(0, \infty)$ with speed g and rate function $J(x) := \lambda - x + x \log(x/\lambda)$, $x > 0$, i.e., for all Borel sets $B \subset (0, \infty)$,

$$\begin{aligned} - \inf_{x \in B^\circ} J(x) &\leq \liminf_{t \rightarrow \infty} \frac{1}{g(t)} \log \mathbb{P}(X(t)/g(t) \in B) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{g(t)} \log \mathbb{P}(X(t)/g(t) \in B) \leq - \inf_{x \in \bar{B}} J(x). \end{aligned}$$

Note that $\{T_C\}_{C \geq 1}$ is the inverse process of $\{X(t)\}_{t \geq 0}$ (see [6] for the formal definition). The claim then follows by Theorem 1 in [6]. Indeed (using the terminology in [6]) the rate function J has no peaks (since $J(\lambda) = 0$) and J decreases on $(0, \lambda)$ and increases on (λ, ∞) . \square

Proof of Proposition 3.2. Note that λ^{-1} is the unique zero of the rate function I in the above lemma. So by a standard application of the large deviations estimates in the lemma and the Borel-Cantelli lemma (see e.g. [5]), we have

$$\lim_{C \rightarrow \infty} \frac{g(T_C)}{C} = 1/\lambda, \quad \text{almost surely.}$$

The claim follows by this relation and the equality $t_C = g^{-1}(C/\lambda)$. \square

Proof of Proposition 3.3 Note that the rate function I in the lemma is continuous, $I(1/\lambda) = 0$ and I decreases on $(0, 1/\lambda)$ and increases on $(1/\lambda, \infty)$. Consequently, for any $x_r > 1/\lambda$, $\inf_{y > x_r} I(y) = \inf_{y \geq x_r} I(y) = I(x_r) = \lambda x_r - 1 - \log(\lambda x_r)$, and, for any $x_l \in (0, 1/\lambda)$, $\inf_{y < x_l} I(y) = \inf_{y \leq x_l} I(y) = I(x_l) = \lambda x_l - 1 - \log(\lambda x_l)$. Relation (3) follows by Lemma B.1 taking $B = (x_r, \infty)$ and relation (4) follows by Lemma B.1 taking $B = (0, x_l)$ (note that g^{-1} is strictly increasing since g is such). \square

APPENDIX C

PROOF OF PROPOSITION 3.4

The proof of this Proposition 3.4 uses Lemma C.1 below, which is of its own interest. We start introducing some notation and definitions. We denote by $\text{Lip}(1)$ the class of real-valued Lipschitz functions from \mathbb{R} to \mathbb{R} with Lipschitz constant less than or equal to one.

Given two real-valued random variables U and Y , the Wasserstein distance between the laws of U and Y , written $d_W(U, Y)$, is defined as

$$d_W(U, Y) := \sup_{\varphi \in \text{Lip}(1)} |\mathbb{E}[\varphi(U)] - \mathbb{E}[\varphi(Y)]|.$$

We recall that the topology induced by d_W on the class of probability measures over \mathbb{R} is finer than the topology of weak convergence (see e.g. [?]).

Lemma C.1: If $g(t) < \infty$, then

$$d_W \left(\frac{X(t) - \lambda g(t)}{\sqrt{\lambda g(t)}}, \mathcal{N}(0, 1) \right) \leq \frac{1}{\sqrt{\lambda g(t)}}.$$

Proof. Define the Borel measure $\mu(dx) := \lambda dg(x)$ over $[0, t]$ (note that g increases on $[0, t]$ and so dg is a Lebesgue-Stieltjes measure) and the function $h(x) = \mathbf{1}_{[0, t]}(x)/\sqrt{\lambda g(t)}$, $x \in [0, t]$. By Corollary 3.4 in [?] and Proposition 3.1, we have

$$\begin{aligned} d_W \left(\frac{X(t) - \lambda g(t)}{\sqrt{\lambda g(t)}}, \mathcal{N}(0, 1) \right) &\leq \\ \left| 1 - \int_{[0, t]} |h(x)|^2 \mu(dx) \right| + \int_{[0, t]} |h(x)|^3 \mu(dx) & \\ = \frac{1}{\sqrt{\lambda g(t)}}. & \end{aligned}$$

\square

Proof of Proposition 3.4. By the assumptions on g we have $C = \lambda g(t_C)$, $t_C \uparrow \infty$ and $g(t_C) \uparrow \infty$, as $C \uparrow \infty$. For any $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(T_C - t_C > x f(t_C)) &= \mathbb{P}(X(t_C + x f(t_C)) < C) \\ &= \mathbb{P} \left(\frac{X(t_C + x f(t_C)) - \lambda g(t_C + x f(t_C))}{\sqrt{\lambda g(t_C + x f(t_C))}} \right. \\ &< \sqrt{\lambda} \frac{g(t_C) - g(t_C + x f(t_C))}{\sqrt{g(t_C + x f(t_C))}} \left. \right). \end{aligned} \quad (18)$$

By Lemma C.1 we have

$$\frac{X(t) - \lambda g(t)}{\sqrt{\lambda g(t)}} \xrightarrow{\text{law}} \mathcal{N}(0, 1), \quad \text{as } t \rightarrow \infty.$$

So, letting C tend to infinity in (18) and using Condition (6) we deduce

$$\lim_{C \rightarrow \infty} \mathbb{P} \left(\frac{T_C - t_C}{f(t_C)} > x \right) = \mathbb{P}(\mathcal{N}(0, 1) \leq -x) = \mathbb{P}(\mathcal{N}(0, 1) > x).$$

The proof is completed. \square

APPENDIX D

PROOF OF PROPOSITION 3.5

We preliminary note that, for any $\delta \in (0, 1)$ and $C > 0$, we have

$$\lambda g(t_C(1 - \delta)) \leq C \leq \lambda g(t_C(1 + \delta)). \quad (19)$$

Indeed, since g is strictly increasing (19) is equivalent to $t_C(1 - \delta) \leq g^{-1}(C/\lambda) \leq t_C(1 + \delta)$, which holds since $t_C = g^{-1}(C/\lambda)$. Note that, due to (8), $p_{\text{hit,Che}}(\cdot)$ is a non-decreasing function. So, for all $\delta \in (0, 1)$, we have

$$\begin{aligned}
& |p_{\text{hit}} - p_{\text{hit,Che}}(t_C)| \leq \\
& \int_{[0, \infty)} |p_{\text{hit,Che}}(\theta) - p_{\text{hit,Che}}(t_C)| \mathbb{P}_{T_C}(\mathrm{d}\theta) \\
& = \int_{[0, t_C(1-\delta)]} (p_{\text{hit,Che}}(t_C) - p_{\text{hit,Che}}(\theta)) \mathbb{P}_{T_C}(\mathrm{d}\theta) \\
& + \int_{(t_C(1+\delta), \infty)} (p_{\text{hit,Che}}(\theta) - p_{\text{hit,Che}}(t_C)) \mathbb{P}_{T_C}(\mathrm{d}\theta) \\
& + \int_{(t_C(1-\delta), t_C(1+\delta)]} |p_{\text{hit,Che}}(\theta) - p_{\text{hit,Che}}(t_C)| \mathbb{P}_{T_C}(\mathrm{d}\theta) \\
& \leq \mathbb{P}(T_C \leq t_C(1 - \delta)) + \mathbb{P}(T_C > t_C(1 + \delta)) \\
& + \max_{\theta \in \{t_C(1-\delta), t_C(1+\delta)\}} |p_{\text{hit,Che}}(\theta) - p_{\text{hit,Che}}(t_C)|.
\end{aligned}$$

The claim follows noticing that by the definition of T_C , the inequality (19) and the properties of the Poisson distribution (see e.g. Lemma 1.2 in [9], formulas (1.10) and (1.11)) we have

$$\begin{aligned}
\mathbb{P}(T_C \leq t_C(1 - \delta)) &= \mathbb{P}(X(t_C(1 - \delta)) > C) \\
&\leq \exp(-\lambda g(t_C(1 - \delta))H(C/\lambda g(t_C(1 - \delta))))
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(T_C > t_C(1 + \delta)) &= \mathbb{P}(X(t_C(1 + \delta)) \leq C) \\
&\leq \exp(-\lambda g(t_C(1 + \delta))H(C/\lambda g(t_C(1 + \delta)))).
\end{aligned}$$

APPENDIX E
AN ILLUSTRATING EXAMPLE

Let h be defined by (5) and assume that Z_1 has support (a, ∞) , $a > 0$.
For $t > 0$, we have

$$\begin{aligned} g(t) &= \int_{-\infty}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} \int_{\max\{-t-u, 0\}}^{-u} \mathbf{1}_{[0, L]}(s) ds} \right] du \\ &= \int_{-\infty}^0 \mathbb{E} \left[1 - e^{-Z_1 \eta(t, u)} \right] du, \end{aligned}$$

where, for $(t, u) \in (0, \infty) \times (-\infty, 0]$,

$$\begin{aligned} \eta(t, u) &:= \frac{1}{L} \int_{\max\{-t-u, 0\}}^{-u} \mathbf{1}_{[0, L]}(s) ds \\ &= \begin{cases} \frac{1}{L} \int_{-t-u}^{-u} \mathbf{1}_{[0, L]}(s) ds & \text{if } -t \geq u \\ \frac{1}{L} \min(L, -u) & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{t}{L} \mathbf{1}_{(-L, 0]}(u) + \frac{t+u+L}{L} \mathbf{1}_{(-L-t, -L]}(u) & \text{if } -t \geq u \\ \frac{1}{L} \min(L, -u) & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, for $t > 0$,

$$\begin{aligned} g(t) &= \int_{-\infty}^{-t} \mathbb{E} \left[1 - e^{-Z_1 \eta(t, u)} \right] du + \int_{-t}^0 \mathbb{E} \left[1 - e^{-Z_1 \eta(t, u)} \right] du \\ &= \int_{-\infty}^{-t} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t \mathbf{1}_{(-L, 0]}(u) + (t+u+L) \mathbf{1}_{(-L-t, -L]}(u))} \right] du + \int_{-t}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} \min(L, -u)} \right] du. \end{aligned}$$

We distinguish two cases: $0 < t \leq L$ and $t > L$. If $0 < t \leq L$, then if $u > -t$ then $-u < L$. So, for $t \in (0, L]$,

$$\begin{aligned} g(t) &= \int_{-\infty}^{-t} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t \mathbf{1}_{(-L, 0]}(u) + (t+u+L) \mathbf{1}_{(-L-t, -L]}(u))} \right] du + \int_{-t}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} u} \right] du \\ &= \int_{(-L-t, -L] \cup (-L, -t]} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t \mathbf{1}_{(-L, 0]}(u) + (t+u+L) \mathbf{1}_{(-L-t, -L]}(u))} \right] du + \int_{-t}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} u} \right] du \\ &= \int_{(-L-t, -L]} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t+u+L)} \right] du + (L-t) \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} t} \right] + t - \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-\frac{Z_1}{L} t} \right) \right] \\ &= 2t + (L-t) \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} t} \right] - 2 \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-\frac{Z_1}{L} t} \right) \right]. \end{aligned}$$

If $t > L$, then if $u \leq -t$ then $u \leq -t < -L$. So, for $t > L$,

$$\begin{aligned} g(t) &= \int_{-\infty}^{-t} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t \mathbf{1}_{(-L, 0]}(u) + (t+u+L) \mathbf{1}_{(-L-t, -L]}(u))} \right] du + \int_{-t}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} \min(L, -u)} \right] du \\ &= \int_{(-L-t, -t]} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t+u+L)} \right] du + (t-L) \mathbb{E} \left[1 - e^{-Z_1} \right] + \int_{-L}^0 \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} u} \right] du \\ &= \int_{(-L-t, -t]} \mathbb{E} \left[1 - e^{-\frac{Z_1}{L} (t+u+L)} \right] du + (t-L) \mathbb{E} \left[1 - e^{-Z_1} \right] + L - \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-Z_1} \right) \right] \\ &= L - \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-Z_1} \right) \right] + (t-L) \mathbb{E} \left[1 - e^{-Z_1} \right] + L - \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-Z_1} \right) \right] \\ &= 2L + (t-L) \mathbb{E} \left[1 - e^{-Z_1} \right] - 2 \mathbb{E} \left[\frac{L}{Z_1} \left(1 - e^{-Z_1} \right) \right]. \end{aligned}$$

Finally, since g is eventually linear with slope $\mathbb{E}[1 - e^{-Z_1}]$ one easily has that Condition (6) is satisfied with

$$f(x) := \sqrt{\frac{x}{\lambda \mathbb{E}[1 - e^{-Z_1}]}}.$$